A UNIQUE DECOMPOSITION THEOREM FOR 3-MANIFOLDS WITH CONNECTED BOUNDARY

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1. **Introduction.** The concern of this paper is the class of triangulated, orientable, connected compact 3-manifolds with connected nonvacuous boundary. If M and M' are two manifolds in this class, one forms their *disk sum* $M \triangle M'$ by pasting a 2-cell on bd (M) to a 2-cell on bd (M'). Up to homeomorphism, the operation of disk sum is well defined, associative, and commutative. One says that a manifold P in this class is *$\triangle$-prime* if P is not a 3-cell, and whenever P is homeomorphic to a disk sum $M \triangle M'$, either M or M' is a 3-cell.

The goal of this paper is the proof of the following theorem, which answers a question of Milnor [3, p. 6] in the affirmative:

Decomposition Theorem. *Let M be a triangulated, orientable, connected compact 3-manifold with connected nonvacuous boundary. If M is not a 3-cell, then M is homeomorphic to a disk sum $P_1 \triangle \cdots \triangle P_n$ of $\triangle$-prime 3-manifolds. The summands P_i are uniquely determined up to order and homeomorphism.*

It will be assumed from now on that any *manifold* is triangulated, orientable, connected and compact. Any map considered is piecewise linear. It will occasionally be convenient to refer to the 3-cell (which is an identity element for the disk sum) or the 3-sphere (which is an identity element for the connected sum—see below) as a *trivial* 3-manifold.

The *connected sum* $M \# M'$ of two 3-manifolds is obtained by removing from each the interior of a 3-cell and then pasting the resulting boundary components together. Up to homeomorphism, the operation of connected sum is well defined, associative, and commutative. One says that a 3-manifold P is *#-prime* if P is not a 3-sphere, and whenever P is homeomorphic to a connected sum $M \# M'$, either M or M' is a 3-sphere.

Definition. Let PCC denote the class of 3-manifolds M with connected nonvacuous boundary such that every 2-sphere in M bounds a 3-cell.

Most of the work of this paper is in proving that the Decomposition Theorem holds for 3-manifolds in the class PCC, which will be done in §3. The lemmas of

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§2 provide major assistance for that proof. The extension of the Decomposition Theorem from 3-manifolds in *PCC* to all 3-manifolds with connected nonvacuous boundary will be accomplished in §4 by an application of Theorems 1 and 2 below. One observes the analogy between the Decomposition Theorem and Theorem 2.

THEOREM 1. *Let M be a* 3-*manifold with connected nonvacuous boundary. Then M is homeomorphic to a connected sum of a* (*possibly trivial*) 3-*manifold in PCC and a* (*possibly trivial*) 3-*manifold with vacuous boundary. The summands are uniquely determined up to homeomorphism.*

The proof of Theorem 1 follows readily from remark 1 of Milnor [3, p. 5].

THEOREM 2. *Let M be a nontrivial* 3-*manifold with vacuous boundary. Then M is homeomorphic to a connected sum $P_1 \# \cdots \# P_n$ of #-prime* 3-*manifolds. The summands P_i are uniquely determined up to order and homeomorphism.*

The present Theorem 2 is Theorem 1 of Milnor [3].

Extension of results. The author uses the methods of this paper to show in [1] that there is a unique decomposition theorem for compact, orientable 3-manifolds with several boundary components. It is also shown in [1] that the problem of classifying the compact, orientable 3-manifolds with several boundary components reduces to the problem of classifying the $\triangle$-prime 3-manifolds.

2. **Some lemmas for Theorem 3.** Let R be a 2-manifold of genus zero whose boundary has $n+1$ components, for some nonnegative integer n, i.e. R is a disk with n holes. If a 3-manifold M is homeomorphic to $R \times [0, 1]$, then M is called a *handlebody of genus n*. In the case that $n=0$, one says M is a *trivial handlebody*. In the case that $n=1$, one say that M is a *handle*.

LEMMA 1. *Let T be the union of a finite family of handlebodies $K_1, \ldots, K_n$ (some of which, perhaps, are trivial) such that the intersection of any K_j with the union of all the other handlebodies K_i is a finite union of disjoint disks on* bd (K_j). *If T is orientable and connected, then T is a* (*possibly trivial*) *handlebody.*

The proof of Lemma 1, which is omitted here, is by an induction on the number n.

LEMMA 2. *Let M be a* 3-*manifold and let D be a disk in* bd (M). *Then there is a* 3-*cell K in M, lying in an arbitrarily small neighborhood of D, such that $K \cap$* bd (M) $=D$. *And* $M \approx \mathrm{cl}\,(M-K)$.

The proof of Lemma 2 is omitted.

LEMMA 3. *Let Y be the union of a finite collection of disjoint* 3-*manifolds and let T be the union of a finite collection of disjoint handlebodies, such that Y and T intersect only on their boundaries, such that the intersection of* bd (T) *with each component of* bd (Y) *is the union of a finite nonvoid family of disjoint disks, and such that $Y \cup T$ is orientable and connected. Then Y contains a family of* 3-*cells, whose union will be denoted by K, such that for $Y' = \mathrm{cl}\,(Y-K)$ and $T' = T \cup K$ the following conditions* *hold*: (i) $Y' \approx Y$. (ii) T' *is the union of a finite collection of disjoint handlebodies.* (iii) Y' *and* T' *intersect only on their boundaries.* (iv) *The intersection of* bd (T') *with each component of* bd (Y') *is a single disk.*

Proof. For each component R_j of bd (Y), let D_j be a disk which contains bd $(T) \cap R_j$ in its interior and let K_j be a 3-cell in Y such that $K_j \cap R_j = D_j$, with the 3-cells K_j chosen so that they are mutually disjoint (see Lemma 2). Let K be the union of the 3-cells K_j. It follows from Lemmas 1 and 2 that conditions (i) and (ii) of the conclusion hold. Conditions (iii) and (iv) of the conclusion evidently follow from the construction.

DEFINITION. Let M and N be manifolds, and let $f: M \to N$ be an imbedding. One says that f is a *proper imbedding* if $f(\text{bd}(M)) \subset \text{bd}(N)$ and $f(\text{int}(M)) \subset \text{int}(N)$.

In Lemma 4 below and thereafter, a *homeomorphism* $(X, U) \to (Y, V)$ will indicate a homeomorphism of X onto Y which carries a subspace U of X onto a subspace V of Y.

LEMMA 4. *Let M be a 3-manifold in the class PCC. Let D be a disk, properly imbedded in M. Let E be another disk, also properly imbedded in M, such that* bd $(D) \cap$ bd (E) *is void and such that* bd $(D) \cup$ bd (E) *is the boundary of an annulus on* bd (M). *Then there is a homeomorphism* $(M, D) \to (M, E)$.

The proof of Lemma 4, which is omitted here, is by a general position argument to reduce the intersection of the disks to a family of simple loops and an induction argument involving cutting at an innermost intersection loop on one of the disks.

LEMMA 5. *Let Y and Y' be homeomorphic 3-manifolds with nonempty boundary, and let C and C' be 3-cells such that* $C \cap Y = \text{bd}(C) \cap \text{bd}(Y)$ *and* $C' \cap Y' = \text{bd}(C') \cap \text{bd}(Y')$, *and such that* bd (C) *intersects each component of* bd (Y) *in a disk and* bd (C') *intersects each component of* bd (Y') *in a disk. If* $Y \cup C$ *and* $Y' \cup C'$ *are orientable, then* $Y \cup C \approx Y' \cup C'$.

The proof of Lemma 5 is omitted.

LEMMA 6. *Let P be a $\triangle$-prime 3-manifold in the class PCC, and let $E_1, \ldots, E_n$ be a collection of disjoint disks, each properly imbedded in P. Let T be the union of the closures of the components of* $P-(E_1 \cup \cdots \cup E_n)$ *which are 3-cells, and let* $Y = \text{cl}(P-T)$ *be nonvoid. Let C be a 3-cell such that C and Y intersect only on their boundaries, such that* bd (C) *intersects each component of* bd (Y) *in a disk, and such that* $C \cup Y$ *is orientable. Then* $Y \cup C \approx P$.

Proof. By Lemma 1, each component of T is a handlebody. Applying Lemma 3 to Y and T, one obtains K, Y', and T' as in the conclusion of Lemma 3. The number of components of T' is the number of components of bd (P), by condition (iv) of Lemma 3. Thus T' is a single handlebody. Choose a disk D on bd (T') such that D contains $T' \cap Y'$ in its interior. By Lemma 2, there is a 3-cell B in the handlebody T' such that $B \cap \text{bd}(T') = D$. Since Y is nonvoid, $Y' \cup B$ is a nontrivial 3-manifold in *PCC*. One observes that $P=(Y' \cup B) \triangle \mathrm{cl}\,(T'-B)$. Since $T' \approx \mathrm{cl}\,(T'-B)$, since P is $\triangle$-prime, and since $Y' \cup B$ is nontrivial, the handlebody T' must be trivial. The conclusion now follows from Lemma 5 because $Y \approx Y'$.

LEMMA 7. *Let P be a $\triangle$-prime 3-manifold in the class PCC and let $E_1, \ldots, E_n$ be a collection of disjoint disks, each properly imbedded in P.* (a) *The closure in P of all but possibly one component of $P-(E_1 \cup \cdots \cup E_n)$ is a 3-cell.* (b) *If the closure in P of every component of $P-(E_1 \cup \cdots \cup E_n)$ is a 3-cell, then P is a handle.*

Proof of (a). Let Y be the union of the closures of the components of $P-(E_1 \cup \cdots \cup E_n)$ which are not 3-cells. Let C be a 3-cell such that C and Y intersect only on their boundaries and $\mathrm{bd}\,(C)$ intersects each component of $\mathrm{bd}\,(Y)$ in a disk, and $Y \cup C$ is orientable. By Lemma 6, $Y \cup C \approx P$. One first considers the case in which Y is not connected. In this case, let D be a disk on $\mathrm{bd}\,(C)$ which contains in its interior all of the components of the intersection of $\mathrm{bd}\,(C)$ with some component of Y. By Lemma 2, there is a 3-cell B in C such that $B \cap \mathrm{bd}\,(C) = D$. The disk $B \cap \mathrm{cl}\,(C-B)$ splits $Y \cup C$ into two nontrivial disk summands, which contradicts the hypothesis that P is $\triangle$-prime. So the case that Y is not connected cannot occur.

Now let Y_1 be the closure of a component of $P-(E_1 \cup \cdots \cup E_n)$ which is not a 3-cell. If $Y \neq Y_1$, then $Y_1 \cap \mathrm{cl}\,(Y-Y_1)$ is the union of a subset of the disks $E_1, \ldots, E_n$. Renumber the disks so that $Y_1 \cap \mathrm{cl}\,(Y-Y_1) = E_1 \cup \cdots \cup E_r$. For $j=1, \ldots, r$, let D_j be a disk in $\mathrm{bd}\,(Y_1)$ which contains E_j in its interior, and let B_j be a 3-cell in Y_1 such that $B_j \cup \mathrm{bd}\,(Y_1) = D_j$ and such that the 3-cells B_j are mutually disjoint (see Lemma 2). For $j=1, \ldots, r$, let E_{n+j} be the disk $B_j \cap \mathrm{cl}\,(Y-B_j)$. One observes that the disks $E_1, \ldots, E_{n+r}$ are disjoint and properly imbedded in P. One also observes that the union Y' of the closures of the components of $P-(E_1 \cup \cdots \cup E_{n+r})$ which are not 3-cells is not connected, for one of its components is $\mathrm{cl}\,(Y_1-(B_1 \cup \cdots \cup B_r))$ and the union of the rest of its components is $\mathrm{cl}\,(Y-Y_1)$. But as above, this contradicts the hypothesis that P is $\triangle$-prime. Therefore $Y_1 = Y$, which completes (a).

Proof of (b). Suppose that the closure in P of every component of $P-(E_1 \cup \cdots \cup E_n)$ is a 3-cell. The intersection of two of these 3-cells is the union of some of the disks E_j. Therefore, these 3-cells satisfy the criteria for the handlebodies of Lemma 1. Hence, P is a handlebody. Since P is $\triangle$-prime, its genus is one.

3. **Decomposition in** *PCC*. This entire section is devoted to the proof of Theorem 3, which is the restriction of the Decomposition Theorem to 3-manifolds in the class *PCC*.

THEOREM 3. *Let M be a nontrivial 3-manifold in the class PCC.* (a) *Then M is homeomorphic to a sum $P_1 \triangle \cdots \triangle P_n$ of $\triangle$-prime 3-manifolds.* (b) *The summands P_i are uniquely determined up to order and homeomorphism.*

Proof of (a). Suppose that M is not already $\triangle$-prime. Then M is homeomorphic to a disk sum $M_1 \triangle M_2$ where M_1 and M_2 are both nontrivial. One observes that a 3-manifold in the class PCC is bounded by a 2-sphere if and only if it is a 3-cell. Hence, the genera of bd (M_1) and bd (M_2) are positive. One also observes that genus (bd (M)) = genus (bd (M_1)) + genus (bd (M_2)). Therefore, the decomposing process terminates in not more than genus (bd (M)) steps, which proves (a).

Proof of (b). Suppose that $M \approx P_1 \triangle \cdots \triangle P_n$, where $P_1, \ldots, P_n$ are $\triangle$-prime 3-manifolds. Then let K be a 3-cell, and let $D_1, \ldots, D_n$ be disjoint disks on bd (K). The 3-manifold M^* is obtained by pasting a disk on bd (P_j) to D_j, for $j=1, \ldots, n$ (see Figure 1). Since $M^* \approx P_1 \triangle \cdots \triangle P_n$, it follows that M^* is homeomorphic to M. It will be convenient to work with M^*.

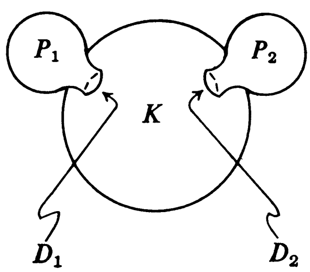

FIGURE 1

Let E be a properly imbedded disk in M^* such that M^*-E has two components and such that the closure of each of these components is nontrivial. The closures of the components of M^*-E will be called M_1 and M_2.

From an induction on the number n of $\triangle$-prime summands P_i of M^*, it follows that (b) of the theorem can be proved by establishing the following statement (which will be done):

A. There is a renumbering of the P_i's such that for some integer r with $1 \leqq r \leqq n$, $M_1 \approx P_1 \triangle \cdots \triangle P_r$ and $M_2 \approx P_{r+1} \triangle \cdots \triangle P_n$.

It will be assumed that the components of $E \cap (D_1 \cup \cdots \cup D_n)$ are simple arcs and simple loops, each a crossing of surfaces, for a general position argument indicates that it suffices to consider this case.

Basis step. Suppose that $E \cap (D_1 \cup \cdots \cup D_n)$ is void. One assumes that the disk E lies in some P_j, for otherwise E would lie in the 3-cell K, which would imply statement A immediately. Surely $P_j - E$ has two components, for otherwise M^*-E would have but one component. Moreover, since P_j is $\triangle$-prime, the closure X of one of the two components of $P_j - E$ is a 3-cell. The disk D_j lies on bd (X), for otherwise X would be the closure of a component of M^*-E, i.e. X would be M_1 or M_2, contradicting their nontriviality. If $X \subset M_1$, then

$$M_2 \approx P_j \quad \text{and} \quad M_1 \approx P_1 \triangle \cdots \triangle P_{j-1} \triangle P_{j+1} \triangle \cdots \triangle P_n.$$

Obviously, a similar decomposition results if $X \subset M_2$. Therefore, statement A holds.

The statements below numbered (3.1), ..., (3.7) are important intermediate results in the proof of the induction step.

Induction step. Let $E \cap (D_1 \cup \cdots \cup D_n)$ have exactly m components.

(3.1) If any component of $E \cap (D_1 \cup \cdots \cup D_n)$ is a loop, then statement A holds.

Proof of (3.1). If any component of $E \cap (D_1 \cup \cdots \cup D_n)$ is a loop, then there is a component k of $E \cap (D_1 \cup \cdots \cup D_n)$ which is a loop and which bounds a subdisk D' of some D_j such that $D' \cap E = k$. Let E' be the subdisk of E which k bounds.

Let E_1 be a disk which lies near but does not intersect the disk $(E-E') \cup D'$, which is properly imbedded in M^*, which lies in general position with respect to $D_1 \cup \cdots \cup D_n$, and which meets $D_1 \cup \cdots \cup D_n$ in fewer than m loops (see Figure 2). By Lemma 4, there is a homeomorphism $f\colon (M^*, E) \to (M^*, E_1)$. Let $M_1' = f(M_1)$ and $M_2' = f(M_2)$. Since the number of components of $E_1 \cap (D_1 \cup \cdots \cup D_n)$ is less than m, there is a renumbering of the P_i's such that $M_1' \approx P_1 \triangle \cdots \triangle P_r$ and $M_2' \approx P_{r+1} \triangle \cdots \triangle P_n$. Hence, $M_1 \approx P_1 \triangle \cdots \triangle P_r$ and $M_2 \approx P_{r+1} \triangle \cdots \triangle P_n$. Thus, statement (3.1) is proved.

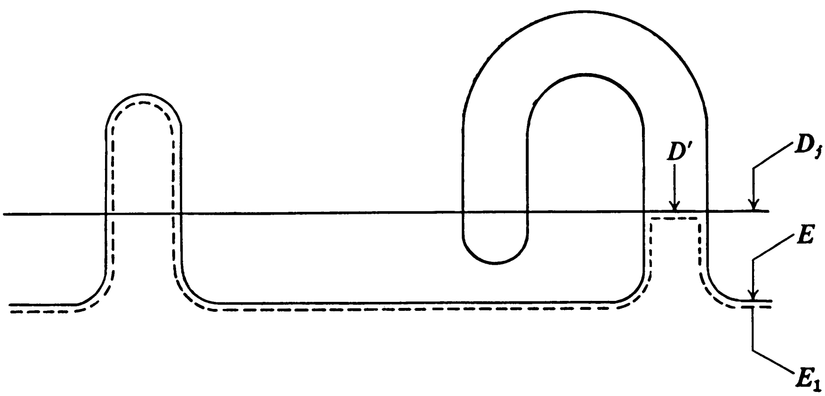

FIGURE 2

It will hereafter be assumed that every component of $E \cap (D_1 \cup \cdots \cup D_n)$ is an arc. This implies, of course, that for $j = 1, \ldots, n$, each component of the intersection of the disk E with the $\triangle$-prime summand P_j is a properly imbedded disk.

DEFINITION. A $\triangle$-prime summand P_j of M^* is called a *special handle* if the closure of every component of $P_j - E$ is a 3-cell. (It follows from (b) of Lemma 7 that a special handle actually is a handle.)

DEFINITION. Let P_j be a $\triangle$-prime summand of M^* such that P_j is not a special handle. *The essence of P_j*, hereafter denoted by Y_j, is the closure in P_j of the component of $P_j - E$ which is not a 3-cell. (It follows from (a) of Lemma 7 that Y_j is well defined.)

One renumbers $P_1, \ldots, P_n$ so that $P_1, \ldots, P_u$ are the summands whose essences lie in M_1, that $P_{u+1}, \ldots, P_s$ are the summands whose essences lie in M_2, and that $P_{s+1}, \ldots, P_n$ are the special handles.

(3.2) Each component of $\text{cl}\,(M_1 - (Y_1 \cup \cdots \cup Y_u))$ is a (possibly trivial) handlebody.

Proof of (3.2). A component of $\text{cl}\,(M_1 - (Y_1 \cup \cdots \cup Y_u))$ is the union of 3-cells, each of which is either the closure of a component of $K - E$ or the closure of a component of some $P_j - E$. Each component of the intersection of any two of these 3-cells is the closure of a component of some $D_i - E$. Therefore, each component of $\text{cl}\,(M_1 - (Y_1 \cup \cdots \cup Y_u))$ satisfies the criteria for the 3-manifold M of Lemma 1. Hence, each is a handlebody.

By an application of Lemma 3 to the union of the essences $Y_1, \ldots, Y_u$ and to $\text{cl}\,(M_1 - (Y_1 \cup \cdots \cup Y_u))$, one now obtains a family of handlebodies, whose union will be called T', and one obtains for each $j = 1, \ldots, u$ a submanifold Y'_j of Y_j which is homeomorphic to Y_j and which satisfies the other properties specified in the proof of Lemma 3.

(3.3) T' is a handlebody (i.e. T' is connected).

Proof of (3.3). Otherwise, the boundary of $M_1 (= \text{bd}\,(T' \cup Y'_1 \cup \cdots \cup Y'_u))$ would not be connected. (See condition (iv) in the conclusion of Lemma 3.)

(3.4) For $j = 1, \ldots, u$, $Y'_j \cap T'$ is the union of a family of disjoint disks, one lying on each component of $\text{bd}\,(Y'_j)$.

Proof of (3.4). This follows from conditions (iii) and (iv) of the conclusion of Lemma 3.

Now let $F_1, \ldots, F_u$ be disjoint disks on $\text{bd}\,(T')$ such that for $j = 1, \ldots, u$, $Y'_j \cap T'$ lies in the interior of F_j. And let $B_1, \ldots, B_u$ be disjoint 3-cells in the handlebody T' such that for $j = 1, \ldots, u$, $B_j \cap \text{bd}\,(T') = F_j$. Define

$$T'' = \text{cl}\,(T' - (B_1 \cup \cdots \cup B_u)).$$

(3.5) For $j = 1, \ldots, u$, $Y'_j \cup B_j \approx P_j$.

Proof of (3.5). This follows immediately from statement (3.4) and Lemma 6.

(3.6) $M_1 \approx P_1 \triangle \cdots \triangle P_u \triangle T''$.

Proof of (3.6). By the construction in the paragraph preceding statement (3.5), $M_1 \approx (Y'_1 \cup B_1) \triangle \cdots \triangle (Y'_u \cup B_u) \triangle T''$. This decomposition of M_1 and statement (3.5) yield statement (3.6) immediately.

Correspondingly, the following statement holds:

(3.7) The 3-manifold M_2 is homeomorphic to the disk sum of $P_{u+1} \triangle \cdots \triangle P_s$ and a (possibly trivial) handlebody U''.

It follows from the Grushko-Neumann theorem (see Kurosh [2, p. 58]) that the sum of genus (T'') and genus (U'') is $n - s$. Therefore, statement A holds.

4. **Proof of the Decomposition Theorem.** Let N be a 3-manifold with vacuous boundary. In this section, N^* denotes the 3-manifold obtained by removing from N the interior of a 3-cell.

LEMMA 8. *Let M be a 3-manifold with connected, nonvacuous boundary, and let N be a 3-manifold with vacuous boundary. Then $M \# N \approx M \triangle N^*$.*

Proof. Let E be the disk in $M \triangle N^*$ across which M and N^* are pasted, and let B be the intersection of N^* and the star of the disk $\operatorname{bd}(N^*) - \operatorname{int}(E)$ in the second barycentric subdivision of $M \triangle N^*$. The 3-manifold obtained by removing $\operatorname{cl}(N^* - B)$ from $M \triangle N^*$ and filling in a 3-cell is homeomorphic to M, and $\operatorname{cl}(N^* - B)$ is homeomorphic to N^*. Therefore $M \triangle N^* \approx M \# N$.

LEMMA 9. *Let M and N be 3-manifolds in the class PCC. Then the disk sum $M \triangle N$ is also in the class PCC.*

Proof. It is obvious that $\operatorname{bd}(M \triangle N)$ is nonvacuous and connected. It will be shown that every 2-sphere in $M \triangle N$ bounds a 3-cell. Let D be the disk in $M \triangle N$ across which M and N are pasted, and let S be a 2-sphere in $M \triangle N$. A general position argument indicates that it suffices to consider the case in which each component of $S \cap D$ is a simple loop in the interior of D at an actual crossing of the surfaces.

Basis step. If $S \cap D$ is empty, then S lies either entirely in M or entirely in N, in which case S bounds a 3-cell in M or N respectively, because the 3-manifolds M and N are in the class PCC.

Induction step. Let $S \cap D$ have n components, $n > 0$. There is a component k of $S \cap D$ which bounds a disk R on the 2-sphere S such that $R \cap D = k$. Let E be the subdisk of D which the loop k bounds. For definiteness, one supposes that the disk R lies in the 3-manifold M. Let U be a neighborhood in M of the disk R and let $g: R \times [-1, 1] \to U$ be a homeomorphism such that the following conditions hold:

(i) For all points x in R, $g(x, 0) = x$.
(ii) $U \cap \operatorname{bd}(M) = g(k \times [-1, 1]) \subset \operatorname{int}(D)$.
(iii) $g(k \times \{-1\}) \subset E$.

The 2-sphere which is the union of the disks $E \cup g(k \times [0, 1])$ and $g(R \times \{1\})$ lies in the 3-manifold M, and since M is the class PCC, it must bound a 3-cell in M. It follows that if D' denotes the disk $(D - (E \cup g(k \times [0, 1])) \cup g(R \times \{1\})$, there is a homeomorphism $(M \triangle N, D) \to (M \triangle N, D')$. Since the number of components of $S \cap D'$ is fewer than n, the 2-sphere S bounds a 3-cell in $M \triangle N$. Hence, $M \triangle N$ is in the class PCC.

Proof of the Decomposition Theorem. Let M be any 3-manifold with connected nonvacuous boundary. By Theorem 1, M is homeomorphic to the connected sum of a 3-manifold P in the class PCC and a 3-manifold Q with vacuous boundary.

Let $P\approx P_1 \triangle \cdots \triangle P_r$ be a $\triangle$-prime decomposition of P, and let $Q\approx Q_1 \# \cdots \# Q_s$ be a #-prime decomposition of Q. Then

$$M \approx P \# Q \approx (P \# Q_1) \# (Q_2 \# \cdots \# Q_s)$$
$$\approx (P \triangle Q_1^*) \# (Q_2 \# \cdots \# Q_s) \approx \cdots \approx P \triangle Q_1^* \triangle \cdots \triangle Q_s^*,$$

by Lemma 8. Therefore, $M\approx P_1 \triangle \cdots \triangle P_r \triangle Q_1^* \triangle \cdots \triangle Q_s^*$, which is a $\triangle$-prime decomposition of M.

Let $M\approx N_1 \triangle \cdots \triangle N_t$ be another $\triangle$-prime decomposition of M, ordered so that $N_{u+1}, \ldots, N_t$ are the summands whose boundaries are 2-spheres. By Lemma 9, the 3-manifold $N_1 \triangle \cdots \triangle N_u$ is in the class PCC. For $j=1, \ldots, t-u$, let V_j be a 3-manifold with vacuous boundary such that $V_j^*=N_{j+u}$. The 3-manifold $V_1 \# \cdots \# V_{t-u}$ has vacuous boundary, and the 3-manifold M is homeomorphic to the connected sum of $N_1 \triangle \cdots \triangle N_u$ and $V_1 \triangle \cdots \triangle V_{t-u}$. By Theorem 1,

$$N_1 \triangle \cdots \triangle N_u \approx P_1 \triangle \cdots \triangle P_r \quad \text{and} \quad Q_1 \# \cdots \# Q_s \approx V_1 \# \cdots \# V_{t-u}.$$

By Theorem 3, $u=r$ and there is a reindexing of the N_i's such that for $i=1, \ldots, r$, $N_i\approx P_i$. By Theorem 2, $s=t-u$ and there is a reindexing of the V_i's such that for $i=1, \ldots, t=u$, $V_i\approx Q_i$, and, therefore, $V_i^*\approx Q_i^*$. The proof of the Decomposition Theorem is now complete.

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